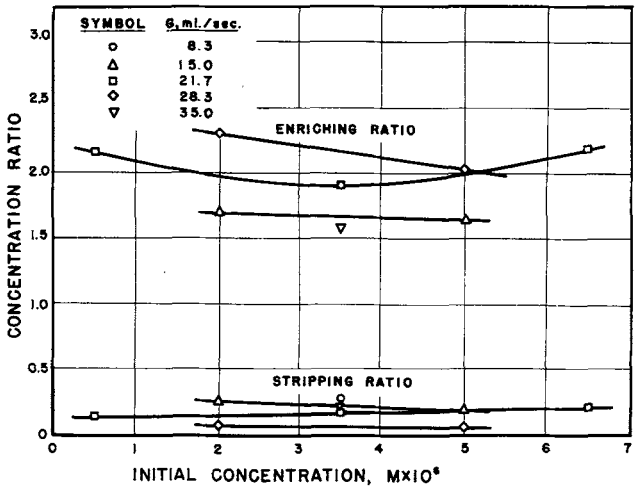

Fig. 2. Overall column performance.

stripping ratio. The latter ratio is defined as the concentration at the bottom of the column divided by the concentration in the charge. Correspondingly, the former is defined as the concentration at the top of the column divided by that in the charge. Dividing the enriching ratio by the stripping ratio yields the overall separation ratio.

Inspection of Figure 2 reveals that the curves are almost horizontally straight lines. This means that variation in the charge concentration was of comparatively little effect on the ratios, including the separation ratio. This agrees with Equations (4), (8), (13), and (14), but, as expected, not with Equations (15) or (16). Of course this comparison with theory is predicated on uniformity of bubble sizes from run to run. Such uniformity is an approximation at best.

Figure 2 also shows that, generally speaking, variation in the gas rate and hence in the bubble frequency resulted in only moderate changes in the ratios. This also supports the general theory, provided bubble sizes are again assumed uniform from run to run. Among the five gas rates employed, the prominent exception was 28.3 ml./sec., especially with respect to its stripping ratio. This however may have been just experimental error since the low concentrations at the bottom of the column are the most difficult to measure accurately.

CONCLUSIONS

Dilute aqueous solutions of crystal violet chloride provide a convenient and effective system for visualizing and studying bubble fractionation. Separation ratios of over 10 have been obtained in some cases, and the separation is clearly visible to the eye.

Experimental column performance is in approximate accord with recent theory. However, more data will be required to distinguish further among the several theoretical submodels. Tests with other dyes would be desirable too.

ACKNOWLEDGMENT

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NOTATION

G = gas rate, ml./sec.
M = concentration in solution, g.-mole/liter
Γ = concentration of adsorbed solute (surface excess), g./sq. cm.

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An Example of Unsteady Laminar Mixing in Power Law Fluids

C. W. VAN ATTA

University of California, San Diego, La Jolla, California

The solution given by Bird (*1*) for the unsteady velocity field produced in a pseudoplastic fluid by a suddenly accelerated moving wall appears to be the only nonsteady analysis for a power law fluid available in the literature. The present note treats the slightly more general problem of the decay of a velocity discontinuity between two parallel streams of fluid, which includes the problem considered in reference *1* as a limiting case. Solutions are obtained for both pseudoplastic and dilatant fluids. The analysis for the pseudoplastic case is nearly formally identical with that of reference *1*. The solution for the dilatant case contains a useful extension of Bird's analysis applicable to dilatant fluids.

We consider the time-dependent, parallel flow produced by momentum transfer between two adjacent laminar streams of fluid. To facilitate comparison with Bird's analysis, the notation of reference *1* will be used wherever appropriate. The initial velocities of the upper and lower streams are U_1 and U_2, respectively. For $t > 0$, the equation of motion for the fluid is (*1*)

$$\rho \frac{\partial u}{\partial t} = mn \left(\frac{\partial u}{\partial y} \right)^{n-1} \frac{\partial^2 u}{\partial y^2} \qquad (1)$$

with boundary conditions

$$u = U_1 \qquad y = +\infty$$
$$u = U_2 \qquad y = -\infty$$

We look for a similarity solution of the form

$$u(y, t) = \tfrac{1}{2}(U_1 + U_2) + \tfrac{1}{2}(U_1 - U_2) f(\zeta)$$

where

$$\zeta = (n+1)^{-1} y \left(\frac{2^{n-1} \rho}{m t V^{n-1}} \right)^{1/(n+1)}$$

and

$$V = U_1 - U_2$$

If a uniform translational velocity equal to $-(U_1 + U_2)$ is imposed on the system so that the new velocity field is $u' = u - (U_1 + U_2)$, the form of Equation (1) is invariant and the boundary conditions become $u' = -U_2$ at $y = +\infty$, $u' = -U_1$ at $y = -\infty$. Hence $u'(-y) = -u(y)$ and $f(-\zeta) = -f(\zeta)$. Since f is antisymmetric about $\zeta = 0$, we may consider only the upper half-plane $y > 0$, for which $(\partial u/\partial y) > 0$. Equation (1) becomes

$$f''(f')^{n-1} + (n+1)^n n^{-1} \zeta f' = 0 \qquad (2)$$

Integrating once, we have

$$f' = \left[\frac{(n+1)^n n^{-1}}{2} (1-n)(B_n + \zeta^2) \right]^{1/(n-1)} \qquad (3)$$

In addition to the free-stream boundary conditions, symmetry requires that $f(0) = 0$. For $n < 1$ the solution satisfying the latter condition is

$$f(\zeta) = \beta_n^{-1/(1-n)} \int_0^\zeta (B_n + w^2)^{-1/(1-n)} dw \qquad (4)$$

where

$$\beta_n = \frac{(1-n)(1+n)^n}{2n}$$

The boundary conditions $f(\infty) = +1$ and $f(-\infty) = -1$ are satisfied if

$$\beta_n^{-1/(1-n)} \int_0^\infty (B_n + w^2)^{-1/(1-n)} dw = 1 \qquad (5)$$

For $n < 1$ (pseudoplastic fluids) the required B_n are the same as those given by Bird. For $n > 1$ (dilatant fluids) the integral is divergent and the solution given by Equation (4) cannot satisfy the boundary conditions at $\zeta = \pm\infty$. To avoid this difficulty, we adopt a procedure analogous to that used for the boundary-layer problem in a power law fluid by Acrivos et al. (2), and change the free-stream boundary condition into the more general form $f = 1$, $f' = 0$ for $\zeta \geq \zeta_c$. This procedure is physically reasonable, inasmuch as it preserves the continuity of the shear stress field. Its usefulness must be determined by comparison of theoretical and experimental velocity profiles.

From the boundary condition $f'(\zeta_c) = 0$ and Equation (3), we find $\zeta_c^2 = -B_n$, and thus for $n > 1$

$$f(\zeta) = (-\beta_n)^{1/(n-1)} \int_0^\zeta (\zeta_c^2 - w^2)^{1/(n-1)} dw \qquad (6)$$

Applying the boundary condition $f(\zeta_c) = 1$ and integrating, we find

$$\zeta_c(n) = (-\beta_n)^{-1/(n+1)} \left[\frac{2\Gamma\left(\frac{3n-1}{2(n-1)}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{n-1}\right)} \right]^{(n-1)/(n+1)} \qquad (7)$$

The mixing region has a finite thickness for $n > 1$, analogous to the finite boundary-layer thickness for $n \geq 2$ found in reference 2.

It is convenient to write the solution in terms of the dimensionless velocity distribution φ_n defined in reference 1 for a pseudoplastic fluid. For $n > 1$, we define φ_n as

$$\varphi_n = (-\beta_n)^{1/(n-1)} \int_\zeta^{\zeta_c} (\zeta_c^2 - w^2)^{1/(n-1)} dw \qquad (8)$$

The dimensionless velocity profile for both dilatant and pseudoplastic fluids may then be written as

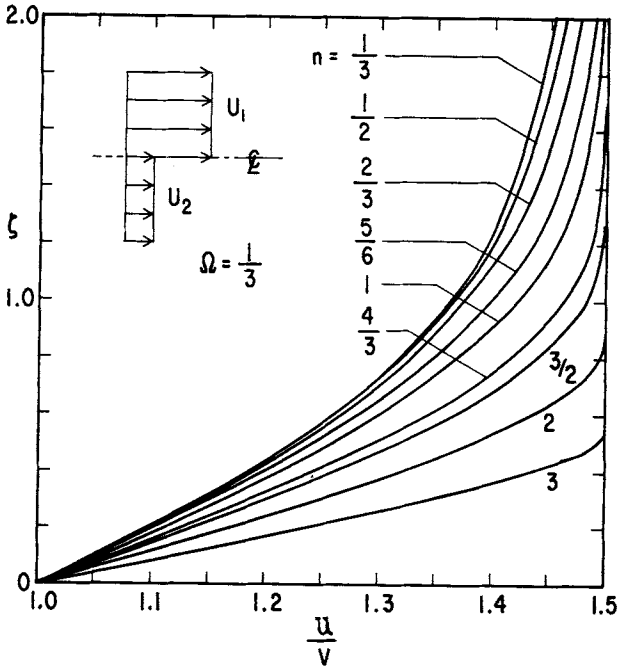

Fig. 1. Dimensionless velocity profiles for unsteady laminar mixing of adjacent streams of power law fluid. Graph shows only upper half of flow, which is antisymmetrical about centerline.

$$u(y,t)/V = \begin{cases} [1/(1-\Omega)] - 1/2\,\varphi_n(\zeta) & \zeta > 0 \\ [\Omega/(1-\Omega)] + 1/2\,\varphi_n(\zeta) & \zeta < 0 \end{cases} \qquad (9)$$

where

$$\Omega = (U_2/U_1)$$

and

$$\varphi_n = \begin{cases} 1-f & \zeta > 0 \\ 1+f & \zeta < 0 \end{cases}$$

In view of the linearity of the boundary conditions, the addition of an appropriate constant to the solution given in reference 1 was to be expected, and for $n < 1$ the form of Equation (9) could have been inferred simply by inspection. The case of a suddenly accelerated flat plate may be recovered by setting the velocity of the plate equal to $V/2$ and taking $\Omega = 0$. The similarity variable and the solution (for $\zeta < 0$) are then identical with those of reference 1. For pseudoplastic fluids, the φ_n given in reference 1 may be used directly in Equation (9). For dilatant fluids, φ_n has been calculated from Equation (8) for several values of n covering the range of practical interest. The expressions for the φ_n (for $\zeta > 0$) and corresponding values of ζ_c from Equation (7) are

$$\varphi_{4/3} = 1 + \frac{1}{8}\left(\frac{7}{3}\right)^{4/3} \left[\frac{\zeta^7}{7} - \frac{3}{5}\zeta_c^2\zeta^5 + \zeta_c^5\zeta^2 - \zeta_c^6\zeta \right]$$

$$\zeta_c(4/3) = 1.680$$

$$\varphi_{3/2} = 1 - \frac{125}{288}\left[\frac{\zeta^5}{5} - \frac{2}{3}\zeta_c^2\zeta^3 + \zeta_c^4\zeta \right]$$

$$\zeta_c(3/2) = 1.340$$

$$\varphi_2 = 1 + \frac{3}{4}\zeta^3 - \frac{9}{4}\zeta_c^2\zeta$$

$$\zeta_c(2) = 0.874$$

$$\varphi_3 = 1 - \frac{8}{2\sqrt{3}}\left[\zeta(\zeta_c^2 - \zeta^2)^{1/2} + \zeta_c^2 \sin^{-1}\frac{\zeta}{\zeta_c} \right]$$

$$\zeta_c(3) = 0.525$$

The resulting dimensionless velocity profiles for both pseudoplastic and dilatant fluids are plotted in Figure 1 for the case $\Omega = 1/3$. For increasing values of the rheological exponent n the effective width of the momentum mixing region decreases, but the approach to the final steady state [in which the velocity is everywhere equal to $1/2\ (U_1 + U_2)$] becomes slower as n increases. For $n > 1$, the velocity profiles approach a linear profile as n increases, a behavior common to steady dilatant flows in pipes and channels.

ACKNOWLEDGMENT

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NOTATION

B_n = constant of integration
f = dimensionless velocity distribution
m,n = parameters characterizing a power law fluid
t = time after mixing begins
u = fluid velocity
U_1 = initial velocity of upper stream
U_2 = initial velocity of lower stream
V = initial velocity difference between two streams
y = distance from centerline

Greek Letters

Γ = gamma function
ζ = dimensionless similarity variable
ρ = fluid density
φ_n = dimensionless velocity distribution as defined in reference *1*
Ω = U_2/U_1 = ratio of initial stream velocities

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On the Stability of a Continuous Flow Stirred-Tank Reactor

DAVID S. SABO and JOSHUA S. DRANOFF

Northwestern University, Evanston, Illinois

The stability of a self-regulating continuous flow stirred-tank reactor (CFSTR) has received considerable attention in recent years. Bilous and Amundson (*1*) presented the first detailed analysis for the case of an exothermic first-order reaction. They demonstrated the multiplicity of possible steady states and analyzed the stability of such states by means of linearization techniques. Berger and Perlmutter (*2, 3*) later applied Lyapunov's second method to obtain a conservative estimate of the region of asymptotic stability for the same problem. Most recently, Paradis and Perlmutter (*4*) used a tracking function technique to determine regions of practical stability and ultimate boundedness. The following development presents a scheme for the simple determination of a general region of ultimate boundedness as well as a set of natural tracking functions for such a system. Application to the special case of an adiabatic reactor system is also demonstrated.

The energy and material balance equations for a CFSTR equipped with a cooling jacket (or coil) are

$$\rho V C_p \frac{dT}{d\theta} = (-\Delta H) V k_o C e^{-E/RT} - A_r U (T - T_c) - \rho q C_p (T - T_o) \quad (1)$$

$$V \frac{dC}{d\theta} = -V k_o C e^{-E/RT} - q (C - C_o) \quad (2)$$

These equations assume first-order, irreversible reaction, constant coolant temperature, and constant physical properties. For convenience, dimensionless variables may be defined and the equations rewritten as follows. Let

$$\bar{T} = T \Big/ \left[\frac{U A_r T_c}{\rho q C_p} + T_o \right] \quad (3)$$

$$\bar{C} = C/C_o \quad (4)$$

$$\bar{\theta} = \theta q/V \quad (5)$$

Then

$$d\bar{T}/d\theta = \alpha \beta \bar{C} e^{-\delta/\bar{T}} - \gamma \bar{T} + 1 \quad (6)$$

and

$$d\bar{C}/d\theta = -\alpha \bar{C} e^{-\delta/\bar{T}} - \bar{C} + 1 \quad (7)$$

where

$$\alpha = k_o V/q \quad (8)$$

$$\beta = (-\Delta H) C_o \Big/ \left[\rho C_p \left\{ \frac{U A_r T_c}{\rho q C_p} + T_o \right\} \right] \quad (9)$$

$$\gamma = \frac{U A_r}{\rho q C_p} + 1 \quad (10)$$

$$\delta = \left[\frac{E}{R} \frac{1}{\left(\frac{U A_r T_c}{\rho q C_p} + T_o \right)} \right] \quad (11)$$

Now, if Equation (7) is multiplied by β and added to Equation (6), one obtains

$$\frac{d(\bar{T} + \beta \bar{C})}{d\theta} = -\left[(\gamma \bar{T} + \beta \bar{C}) - (1 + \beta) \right] \quad (12)$$

This equation provides the key to further analysis. When the reactor system is at steady state, the left-hand side of Equation (12) is equal to zero, so that the steady state (or singular) points of Equations (6) and (7) must lie on a straight line—the steady state line—given by

$$\gamma \bar{T} + \beta \bar{C} = 1 + \beta \quad (13)$$

Figure 1 shows a phase plane with this readily determined steady state locus indicated as a dotted line. Also shown are contours of constant $(\bar{T} + \beta \bar{C})$. All of these lines